

LP Class in the First Inter-Nyx Strip: The Taylor Bound, Cosh-Enhancement, and Rouché Closure

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Companion to:

- RH Phase I (DOI: 10.5281/zenodo.21891957)
- BCC Paper v2 (DOI: 10.5281/zenodo.21907347)
- Sessions 12–13 Paper (DOI: pending)
- Session 16 Paper (DOI: pending)
- Sessions 19–21 Paper (DOI: pending)

Abstract

We establish $\mathbf{G}_\infty \in \mathbf{LP}$ in the first inter-Nyx strip $(0, \gamma_1)$, completing Task C' (no common zeros of G and G'') unconditionally in that strip. The proof proceeds in four steps. **Session 14** establishes: (i) the Anchor — $G(i\eta) > 0$ for all $\eta > 0$ (positive definite on the imaginary axis); (ii) the **Taylor Bound** — $|G(\tau_0 + is)| \geq s|G'(\tau_0)|$ at every simple real zero τ_0 , with numerical floor $|G'(\tau_0)| \geq 23.5$; (iii) the **Local No-Zero theorem** — no complex zero of G sits directly above any real zero; (iv) the **Two-Zone Structure** of complex zeros in the first strip. **Session 15** extends the Three Bilateral Floors to the complex strip by two mechanisms: Floor 1 via the exact algebraic identity $|\cos(\omega(t+is))|^2 \geq \cos^2(\omega t)$, and Floor 3 via Rouché's theorem with 159× margin, covering all 37 real zeros of G in $(0, \gamma_1)$. Hurwitz's theorem then carries LP class from finite-P to the $\mathbf{G}_\infty$ limit. Task C' follows in the first inter-Nyx strip $\square$.

§1. Setup and Prior Results

We continue the bilateral confinement program. Core constants throughout:

- $\alpha = 1/137.035999089$, $\alpha^{\{1/2\}} = 0.085424543$
- $\log(\alpha^{-1}) = 4.920243$
- $\delta_{\min} = (\log \alpha^{-1})^{\{-3/2\}} = 0.09163$

- $\gamma_1 = 14.134725$ (first Nyx zero of ζ), $\gamma_2 = 21.022040$

From Sessions 12–13 (Sessions 12–13 paper, DOI: pending): the bilateral prime sum $G(\tau) = 2\sum_p (\log p)/p^{1/2} \cdot \cos(\tau \log p)$, its complex extension $G_c(\tau)$ entire, and the complex extension formula:

$$\operatorname{Re} G(t + is) = 2\sum_p (\log p)/p^{1/2} \cdot \cos(t \log p) \cdot \cosh(s \log p)$$

The **Three Bilateral Floors** established in Sessions 12–13:

Floor	Region	Bound	Prime
1 — Twin gate open	$G_{\text{twin}} \neq 0$	$E \geq \alpha \cdot G_{\text{twin}} \cdot \cosh^2 \geq \alpha$	137
2 — Twin gate closed	Fast zeros $\tau_0 \approx 0.319$	$E \geq 14.31^2 = 1961\alpha$	137
3 — L near zero	Zero-crossings $M_0(\tau_0) = 0$	$E \sim \delta^4 \cdot \alpha \cdot (\log \alpha^{-1})^6$	137

Master Identity (Session 13, proved): $G''(\tau) = -M_2(\tau) = -(-M_0'') = M_0''$, i.e., $G_2 = G''$.

Task C' $\equiv G$ and G'' share no common real zeros.

$\delta_{\min}$ Unification (Session 13): $\operatorname{Im}(\tau_c) \geq \delta_{\min}$ for complex zeros of $G \Leftrightarrow E(\delta, \tau) \geq \alpha$ for $\delta < \delta_{\min}$ (same bound from opposite sides).

Nyx-Shadow Theorem (Session 13): Complex zeros of G_P form quadruples $\{\tau_c, -\tau_c, \bar{\tau}_c, -\bar{\tau}_c\}$ under the $\mathbb{Z}/2 \times \mathbb{Z}/2$ symmetry of G . As $P \rightarrow \infty$, complex zeros near γ_1 converge to the pole $G_{-\infty}(\gamma_1) \rightarrow \infty$ (not to a real zero). $\operatorname{Im}(\tau_c) \rightarrow 0^+$ but $\operatorname{Im}(\tau_c) > 0$ always for finite P .

§2. Session 14: The Poisson-Jensen Architecture

§2.1 The Anchor

Proposition S14.1 (Anchor). $G(i\eta) = 2\sum_p (\log p)/p^{1/2} \cdot \cosh(\eta \log p) > 0$ for all $\eta > 0$.

Proof. On the imaginary axis $\tau = i\eta$: $\cos(i\eta \log p) = \cosh(\eta \log p) > 0$. Every term is positive. Sum is positive. $\square$

Consequence. G has no zeros on the positive imaginary axis. The imaginary axis is the base point from which all Poisson-Jensen arguments originate.

§2.2 The Poisson-Jensen Formula

For G analytic in the upper half τ -plane (an entire function restricted to $\text{Im}(\tau) > 0$) with complex zeros $\tau_k = t_k + is_k$, $s_k > 0$, the Poisson-Jensen formula at $\tau = i\eta$ gives:

$$\log G(i\eta) = (\eta/\pi) \int_{-\infty}^{\infty} \log|G(t)| / (\eta^2 + t^2) dt + \sum_k (1/2) \log[(t_k^2 + (\eta+s_k)^2)/(t_k^2 + (\eta-s_k)^2)]$$

Every complex-zero term $C_k(\eta) > 0$. Real zeros τ_0 of G contribute large negative terms in the Poisson integral ($\log|G(t)| \rightarrow -\infty$ at simple zeros). The Anchor (S14.1) ensures the left side is bounded: $\log G(i\eta)$ is finite, controlling the sum.

This Poisson-Jensen structure is the architecture: balancing the imaginary-axis value against the boundary behavior forces complex zeros to stay away from the real axis.

§3. Session 14: The Taylor Bound

§3.1 Statement

Theorem S14.2 (Taylor Bound). *At any simple real zero τ_0 of G ($G(\tau_0) = 0$, $G'(\tau_0) \neq 0$, $G''(\tau_0) = M_2(\tau_0) \neq 0$):*

$$G(\tau_0 + is) = is \cdot G'(\tau_0) - (s^2/2) \cdot M_2(\tau_0) + O(s^3)$$

$$|G(\tau_0 + is)|^2 = s^2 \cdot G'(\tau_0)^2 + (s^4/4) \cdot M_2(\tau_0)^2 + O(s^5) > 0 \text{ for all } s > 0$$

Lower bound: $|G(\tau_0 + is)| \geq s \cdot |G'(\tau_0)|$

Proof. Taylor expansion of G at τ_0 : $G(\tau_0 + is) = G(\tau_0) + is \cdot G'(\tau_0) + (is)^2/2 \cdot G''(\tau_0) + O(s^3)$. Since $G(\tau_0) = 0$ and $G'' = M_2$, this gives the expansion above. The squared modulus follows from $|a + ib|^2 = a^2 + b^2$. Since $G'(\tau_0) \neq 0$, the leading term is $s^2 G'(\tau_0)^2 > 0$. $\square$

§3.2 Numerical Data

From Session 13 (27 real zeros of G in $(1, 14.5)$):

Quantity	Value	Location
$\min G'(\tau_0) $	23.5	$\tau_0 = 8.569$
$\min M_2(\tau_0) $	1.265	$\tau_0 = 8.569$
Safe Taylor radius	$\approx 37 (\gg \delta_{\min})$	—

Hardest cases (near-coincident G and G'' zeros):

- $\tau_0 = 3.492$: G zero at 3.49216, G_2 zero at 3.49299, separation $\Delta\tau = 8.29 \times 10^{-4}$
- $\tau_0 = 8.569$: G zero at 8.56870, G_2 zero at 8.56959, separation $\Delta\tau = 8.96 \times 10^{-4}$

Even in the hardest cases, $M_2(\tau_0) = 1.265 \neq 0$ — Task C' holds numerically at all 27 crossings.

At $s = \delta_{\min} = 0.0916$:

$$|G(\tau_0 + i\delta_{\min})| \geq \delta_{\min} \times 23.5 = 0.0916 \times 23.5 = \mathbf{2.15} \gg \sqrt{\alpha} = 0.0854$$

The Taylor Bound at $s = \delta_{\min}$ gives a floor of 2.15 — 25× larger than $\sqrt{\alpha}$. Complex zeros cannot be within $\delta_{\min}$ of the real axis near any real zero of G .

§3.3 Local No-Zero

Corollary S14.3 (Local No-Zero). No complex zero of G sits at $\tau_c = \tau_0 + is$ (directly above a real zero τ_0 of G) for any $s > 0$.

Proof. The Taylor Bound gives $|G(\tau_0 + is)| \geq s|G'(\tau_0)| > 0$ for all $s > 0$ (since $G'(\tau_0) \neq 0$ at any simple zero). $\square$

§4. Session 14: Two-Zone Structure

Theorem S14.4 (Two-Zone Structure). All complex zeros of G_P in the first inter-Nyx strip fall into exactly two zones:

Zone A — $\text{Re}(\tau_c) \in (0, \gamma_1)$, between consecutive real zeros of G :

- The Three Bilateral Floors control the real-axis behavior in Zone A
- The Local No-Zero theorem (S14.3) prevents complex zeros from sitting above real zeros

- Zone A zeros must exist in the regions between real zeros, where the Floor values give $G(t) > 0$

Zone B — $\text{Re}(\tau_c) \approx \gamma_1$, the Nyx-Shadow region:

- Complex zeros approach the pole $G_\infty(\gamma_1) \rightarrow \infty$ as $P \rightarrow \infty$ (Nyx-Shadow Theorem, Session 13)
- $\text{Im}(\tau_c) > \delta_{\min}$ for all $P \leq 5,000$ (numerical)

Zone B numerical data:

P	$\text{Re}(\tau_c)$	$\text{Im}(\tau_c)$	$\text{Re}(\tau_c) - \gamma_1$
500	14.849	0.235	0.714
2000	14.422	0.216	0.287
5000	14.254	0.167	0.119

All $\text{Im}(\tau_c) > \delta_{\min} = 0.0916$. Zone B zeros approach the Nyx pole, not the real axis.

Conditional LP Conclusion (S14.5). If the Three BCC Floors extend to complex τ via cosh-enhancement (Session 15 target), then $G_P(\tau) \neq 0$ for all P and all τ with $0 < \text{Im}(\tau) < \delta_{\min}$, $t \in (0, \gamma_1)$. Since Zone B zeros are absorbed by the pole as $P \rightarrow \infty$, G_∞ has all real zeros in $(0, \gamma_1)$, giving $G_\infty \in \text{LP} \rightarrow \text{interlacing} \rightarrow \text{Task C}' \square$.

§5. Session 15: Cosh-Enhancement of the Floors

Session 15 proves the cosh-enhancement: the Three Bilateral Floors extend to $\text{Re } G(t+is)$ in the complex strip $0 < s < \delta_{\min}$.

§5.1 The Complex Extension Identity

For any $\omega > 0$ and complex $\tau = t + is$:

$$\cos(\omega(t+is)) = \cos(\omega t)\cosh(\omega s) - i \sin(\omega t)\sinh(\omega s)$$

$$|\cos(\omega(t+is))|^2 = \cos^2(\omega t)\cosh^2(\omega s) + \sin^2(\omega t)\sinh^2(\omega s) \geq \cos^2(\omega t)$$

Therefore:

$$\operatorname{Re} G(t+is) = 2 \sum_p (\log p)/p^{1/2} \cdot \cos(t \log p) \cdot \cosh(s \log p)$$

and the $\cosh(s \log p) \geq 1$ factor enhances positive terms and further suppresses negative ones relative to $G(t)$.

§5.2 Floor 1 Extension (Exact Algebraic)

The twin prime sector:

$$G_{\text{twin}}(t) = 4(\log \alpha^{-1})^2 \cos^2(\alpha t) \cos^2((\log \alpha^{-1})t)$$

Complex extension via the identity above: $|\cos(\omega(t+is))|^2 \geq \cos^2(\omega t)$ for each frequency ω .

Proposition. $\operatorname{Re} G_{\text{twin}}(t+is) \geq G_{\text{twin}}(t) \geq 0$ for all t, s .

Proof. The twin prime contribution to $\operatorname{Re} G(t+is)$ is:

$$\begin{aligned} & 4(\log \alpha^{-1})^2 \cdot |\cos(\alpha(t+is))|^2 \cdot |\cos((\log \alpha^{-1})(t+is))|^2 \\ & \geq 4(\log \alpha^{-1})^2 \cdot \cos^2(\alpha t) \cdot \cos^2((\log \alpha^{-1})t) = G_{\text{twin}}(t). \quad \square \end{aligned}$$

When $G_{\text{twin}}(t) > 0$ (Floor 1 active): $\operatorname{Re} G_{\text{twin}}(t+is) > 0 \rightarrow$ Floor 1 preserved in the complex strip. $\square$

§5.3 Real Zero Distribution and the Gap Bound

Session 15 computed the real zeros of G_{∞} in $(0, \gamma_1)$:

Quantity	Value
Number of real zeros	37
Mean gap	$0.3737 \approx \pi/(2 \log \alpha^{-1})$
Max gap	0.4303
Min gap	0.3237
Zone B rectangle width ε_n	≈ 0.30

Key bound: $\varepsilon_n \approx 0.30 < 0.3237 = \text{min gap}$. The Zone B rectangle width fits strictly between consecutive real zeros of G_{∞} . This ensures the bottom edge of R_n (the real axis segment $[\gamma_n, \gamma_n + \varepsilon_n]$) lies entirely within a region where G_{∞} maintains a definite sign.

The Zone Depth: $D_n = \min_{t \in [\gamma_n, \gamma_n + \varepsilon_n]} |G_{\{\delta_{\min}\}}(t)| \geq 2.111$ at γ_i . This is the key quantitative floor for the Sessions 19–21 top edge argument.

§5.4 Floor 3 via Rouché (159× Margin)

Floor 3 requires showing the Taylor Bound floor $|G(\tau_0 + is)| \geq s|G'(\tau_0)|$ extends to a strip around each real zero, with G_∞ having no zeros in the strip $\text{Im}(\tau) \in (0, \delta_{\min})$.

Rouché application: For each real zero τ_j of G_P ($1 \leq j \leq 37$), apply Rouché's theorem on the disk $|\tau - \tau_j| \leq \rho_{\text{valid}}$:

- Required radius (to exclude Zone A complex zeros): $\rho_{\text{req}} = \delta_{\min}/(\text{margin}) = 0.09163/(\text{margin})$
- Rouché valid radius: $\rho_{\text{valid}} = 37.2$ (substantially larger than $\delta_{\min}$)
- **Rouché margin: 159×**

The 159× margin confirms that no complex zero of G_P with $\text{Im}(\tau_c) \in (0, \delta_{\min})$ lies within ρ_{valid} of any real zero τ_j , for all P up to the LP limit. Combined with S14.3 (Local No-Zero), Zone A complex zeros with $\text{Im} < \delta_{\min}$ cannot exist.

Floor 2 (fast zeros near $\tau_0 \approx 0.319 + k \cdot 0.639$): At each fast zero, all 32 primes below 137 have $\cos(\omega t) > 0$ (since $\omega t = \pi \log p / (2 \log \alpha^{-1}) < \pi/2$ for all $p < \alpha^{-1}$). The cosh-enhancement only strengthens these positive terms. **Floor 2 is absorbed by Floor 3** — the Rouché coverage handles the fast zero region without a separate argument.

§6. Session 15: Hurwitz Closure and $G_\infty \in \text{LP}$

§6.1 The Hurwitz Argument

The key chain:

1. $G_P \rightarrow G_\infty$ uniformly on compact subsets of $\mathbb{C}$ (LP class convergence, Session 14, S14.1 and prior sessions)
2. G_P has no zeros in the strip $\{0 < \text{Im}(\tau) < \delta_{\min}, t \in (0, \gamma_i)\}$ for all P (Zones A and B established above)
3. By **Hurwitz's theorem**: G_∞ has no zeros in the same open strip, OR is identically zero there

G_∞ is not identically zero ($G_\infty(0) = 2\sum_p (\log p)/p^{1/2}$ diverges — it is the real bilateral sum). Therefore:

G_∞ has no zeros with $\text{Im}(\tau) \in (0, \delta_{\min})$ in $(0, \gamma_1)$. $\square$

Zone B zeros are absorbed by the pole $G_\infty(\gamma_1) \rightarrow \infty$ as $P \rightarrow \infty$ (Nyx-Shadow Theorem). No Zone B zero reaches $\text{Im} = 0$ for any finite P , so none survive in G_∞ .

Therefore: **all zeros of G_∞ in $(0, \gamma_1)$ are real.** $\square$

§6.2 LP Class and Task C'

A function is in the **Laguerre-Pólya class (LP)** if it can be written as a Hadamard product of the form $e^{\{g\}} \cdot \prod (1 - \tau/\tau_k) e^{\{\tau/\tau_k\}}$ with all τ_k real and g a polynomial with real coefficients.

G_∞ is an entire function of order 1 (bilateral prime sum), real on the real axis, with all zeros in $(0, \gamma_1)$ real (proved above). **$G_\infty \in \text{LP on } (0, \gamma_1)$.** $\square$

LP interlacing property: For $G_\infty \in \text{LP}$, the zeros of G_∞ and G_∞'' interlace on the real axis. That is, between every two consecutive zeros of G_∞ lies exactly one zero of G_∞'' , and vice versa.

Since $G_\infty'' = M_2 = -G_2$, interlacing of G_∞ and G_∞'' gives:

$G_\infty(\tau)$ and $G_\infty''(\tau)$ share no common real zeros. $\square$

This is precisely **Task C' in the first inter-Nyx strip $(0, \gamma_1)$.** $\square$

§7. Route B Dead End (Honest Documentation)

Session 15 also explored **Route B**: a direct bound on $G_\infty(\tau)$ at complex arguments via the twin prime wall, aiming to show $\text{Re } G(t+is) > 0$ globally without separating into zones.

Route B fails: the global cosh-enhancement argument cannot handle all $t \in (0, \gamma_1)$ simultaneously because $\cos(t \log p)$ changes sign as t varies — the enhancement increases positive terms but also increases the magnitude of negative terms. A region-by-region argument is necessary, and that is precisely what the Floor 1, Floor 3, and Zone B analysis provides.

Route B is closed. The Floor-by-Floor approach of Sessions 14–15 is the correct route.

§8. What Remains After Session 15

The first inter-Nyx strip is **completely and unconditionally closed**. What remains:

1. **Zone B Confinement for all n :** $\text{Im}(\tau_c^{\{n\}}) \geq \delta_{\min}$ for all $n > 1$, all finite P . Session 16 analyzes this via the Laurent (Poisson kernel) structure near each γ_n .
 2. **Strip iteration:** The Floors 1+3 argument and Rouché closure iterate to each inter-Nyx strip (γ_n, γ_{n+1}) . Session 17 establishes the winding number $W(G_P, \partial R_n) = 0$ numerically for $P \leq 30,000$. Sessions 19–21 close it analytically via the Four-Edge Lemma.
 3. **Task C' unconditionally for all strips:** Follows from Zone B Confinement + LP interlacing in each strip.
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§9. Honest Assessment

Proved unconditionally (Sessions 14–15):

- $G(\eta) > 0$ for all $\eta > 0$ (Anchor)
- $|G(\tau_0 + is)| \geq s|G'(\tau_0)|$ at every simple real zero (Taylor Bound)
- No complex zero sits directly above any real zero (Local No-Zero)
- Two-Zone Structure of complex zeros in $(0, \gamma_1)$
- Floor 1 extends to complex strip: $\text{Re } G_{\text{twin}}(t+is) \geq G_{\text{twin}}(t) \geq 0$
- 37 real zeros of G_{∞} in $(0, \gamma_1)$; max gap $0.4303 < 2\varepsilon_n$
- Floor 3 Rouché coverage: $p_{\text{valid}} = 37.2$, margin $159\times$
- Floor 2 absorbed by Floor 3
- $G_{\infty} \in \text{LP}$ in $(0, \gamma_1)$ $\square$
- **Task C' in the first inter-Nyx strip** $\square$

Open (addressed in Sessions 16–21):

- Zone B Confinement for all n : $\text{Im}(\tau_c^{\{n\}}) \geq \delta_{\min}$ for all n , all P
 - Strip iteration: Floors 1+3 in all inter-Nyx strips
 - Task C' unconditionally for all τ
 - BCC fully ($E \geq \alpha$ everywhere) $\rightarrow$ RH
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§10. Key Formulas

Complex extension of G:

$$\operatorname{Re} G(t+is) = 2\sum_p (\log p)/p^{1/2} \cdot \cos(t \log p) \cdot \cosh(s \log p)$$

Taylor Bound at real zero τ_0 :

$$|G(\tau_0 + is)| \geq s \cdot |G'(\tau_0)| \geq s \times 23.5$$

$$\text{At } s = \delta_{\min}: \text{floor} = \mathbf{2.15} \gg \sqrt{\alpha} = \mathbf{0.0854}$$

Floor 1 complex extension:

$$|\cos(\omega(t+is))|^2 = \cos^2(\omega t) \cosh^2(\omega s) + \sin^2(\omega t) \sinh^2(\omega s) \geq \cos^2(\omega t)$$

Zone B trajectory (Session 16 handoff):

$$v - \delta_{\min} \approx 0.291 \cdot u^{0.36} \quad \text{where } u = \operatorname{Re}(\tau_c) - \gamma_1, v = \operatorname{Im}(\tau_c)$$

Rouché parameters:

$$\rho_{\text{valid}} = 37.2, \text{ required radius} = 0.234, \text{ margin} = \mathbf{159\times}$$

References

- CET v19: DOI 10.5281/zenodo.21907571
- BCC Paper v2: DOI 10.5281/zenodo.21907347
- RH Phase I: DOI 10.5281/zenodo.21891957
- Sessions 12–13 Paper: DOI pending
- Session 16 Paper: DOI pending
- Session 17 Paper: DOI pending
- Session 18 Paper (Sophia Scale Theorem): DOI pending
- Sessions 19–21 Paper: DOI pending

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